0906 Lecture 01

Sunday, September 04, 2016 9:17 PM

Outline:

1) Moduli problems in algebraic geometry
2) Equivariant geometry and geometric invariant theory
3) The moduli of vector bundles on a curve
4) Atiyah-Bott formula for the Betti numbers, and the Verlinde formula
5) Stratifications - Kirwan surjectivity

1 Moduli problems in algebraic geometry

Guiding meta-problem in mathematics is the problem of <u>classification</u>: Famous example, classification of simple Lie algebras over $\mathbb{C}$ in terms of their Dynkin diagrams $(A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2)$

In algebraic geometry, often classification involves finitely many non-discrete parameters.

Ex: (Riemann) moduli of holomorphic structures on a smooth oriented topological surface. Riemann predicted there are $3g-3$ complex parameters, developed by Ahlfors & Bers

- algebro-geometric point of view:
  Infinitesimal deformations of the complex structure on a curve are classified by

$$\underbrace{H^1(X, TX)}_{\substack{\text{general fact} \\ \text{for} \\ \text{cplx manifolds}}} = \underbrace{H^1(C, K^\vee)}_{\substack{\text{when} \\ X=C \text{ is} \\ \text{a curve}}} \underset{\text{Serre duality}}{\cong} H^0(C, K^{\otimes 2})^\vee$$

$g>1$: $H^1(C, K^{\otimes 2}) = 0$, so Riemann-Roch gives

$$\dim H^0(C, K^{\otimes 2}) = \deg(K^{\otimes 2}) + (1-g) = 3(g-1)$$

(Remark: there are several ways to see this, ranging

(Remark: there are several ways to see this, ranging from the Kodaira-Spencer map, more algebro-geometric, to a direct identification between first-order def of integrable almost complex structures and Dolbeault cohomology cocycles in $H^1(X, TX)$.)

- diff. geom. point of view: S smooth surface of genus > 1

$H(S)$ = space of complete Riem. metrics, const. curv. -1

Diff_0 = diffeom. isotopic to identity

$\Gamma = \mathrm{Diff}/\mathrm{Diff}_0$ = mapping class group

$H(S)/\mathrm{Diff}_0$ = Teichmuller space $\cong \mathbb{R}^{6(g-1)}$, has canonical complex structure

$\phantom{H(S)/\mathrm{Diff}_0 = \text{Teichmuller space}} \; \| \; T(S)$

$M_g = T(S)/\Gamma$ = moduli space of genus g-curves

In fact, M_g canonically has the structure of a projective variety (with local quotient singularities).

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↳ <u>Rem</u>: useful for enumerative questions!

$\boxed{\text{Goal}}$: to have a general framework for studying moduli problems in AG, finding/constructing "moduli spaces"

② <u>Equivariant geometry</u>

The most concrete method for constructing moduli spaces is via equivariant AG.

<u>Context</u>: 1) G = reductive group $/\mathbb{C}$ (i.e. complexif. of compact Lie group)

2) linear action of G on $\mathbb{P}^n$

(in fact every algebraic action is linear, because (in fact every algebraic action is linear, because $\mathrm{Aut}(\mathbb{P}^n) \cong PGL_{n+1}$, which can be seen from functor-of-points definition of $\mathbb{P}^n$)

3) $X \hookrightarrow \mathbb{P}^n$ a locally closed subvariety equivariant for action of G

We will develop a theory of equivariant cohomology, K-theory, coherent & quasi-coherent sheaves, etc.

$\boxed{\text{Guiding principle:}}$ Any equivariant notion should be independent of the specific description of X/G as a quotient:

- If G acts freely on X

E.g. if G acts freely on X in an appropriate sense, then there is a scheme X/G parameterizing G-orbits in X, and the equivariant cohomology $H^*_G(X) \cong H^*(\text{space } X/G)$

In an ideal world, we could form a quotient space X/G – an algebraic variety whose points classify orbits

↳ Not possible, e.g. for $\mathbb{C}^n/\mathbb{C}^*$ any map $\mathbb{C}^n/\mathbb{C}^* \to X$ factors through $\mathbb{C}^n/\mathbb{C}^* \longrightarrow pt$, which follows from the fact that all invariant functions are constant

Nevertheless we can regard X/G as a kind of geometric object $\rightsquigarrow$ quotient stack

geometry on X/G $\longleftrightarrow$ equivariant geometry on X

We will discuss several perspectives, tradeoff between abstraction and "cleanliness"

equivariant geometry $G \circlearrowright X$ $\rightsquigarrow$ geometry of groupoid schemes $\rightsquigarrow$ geometry of algebraic stacks

Why we care about equivariant geometry:

Ex: There is a quasi-projective scheme $X_{g,d,n}$ constructed using "Hilbert schemes" which parameterizes smooth curves using "Hilbert schemes" which parameterizes smooth curves $C \hookrightarrow \mathbb{P}^n$ and such that the action of $PGL_{n+1} \curvearrowright \mathbb{P}^n$ extends to $X_{g,d,n}$. The action is semi-free (meaning points have finite stabilizers). There <u>is</u> a quotient space in this case $\rightsquigarrow$ it is M_g!

③ Moduli of vector bundles over a curve

This will be our main example. It is of fundamental interest in the geometric Langlands program, but we will mostly study it because it a beautiful example exhibiting so much $\boxed{\text{pathology}}$ yet having so much structure.

much structure.

1) it is highly non-separated
2) there are too many vector bundles to be parameterized by a single scheme

(will explain what these both mean later in course)

Can do the same kind of computation $\rightsquigarrow$ 1st order deformations of $\mathcal{E}$ are classified by

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \rightsquigarrow \dim = (n^2-1)(g-1) \text{ for } \mathcal{E} \text{ generic}$$

So vector bundles specified by continuous param.'s.

⋮

There is an algebraic stack $\mathcal{M}_{n,d}(C)$ parameterizing

There is an <u>algebraic stack</u> $\mathcal{M}_{n,d}(C)$ parameterizing rank n degree d vector bundles over C. We will discuss several constructions:

- → functor-of-points description (quickest)
- → local quotient description
- → global quotient description in the algebraic category using infinite Grassmannians
- → global quotient description in terms of mathematical gauge thy (Atiyah-Bott)

The stack $\mathcal{M}_{n,d}(C) = \mathcal{M}_{n,d}$ has the pathologies 1&2 above. But it has a special $\underbrace{\text{stratification}}_{\text{Harder-Narasimhan-Shatz}}$

$$\mathcal{M}_{n,d} = \mathcal{M}_{n,d}^{ss} \cup \bigcup S_\alpha$$

$$\mathcal{M}_{n,d} = \mathcal{M}_{n,d}^{ss} \cup \bigcup_{\alpha} S_{\alpha}$$ Narasimhan -Shatz

where α ranges over all matrices $\alpha = \begin{bmatrix} d_1 & \cdots & d_k \\ n_1 & & n_k \end{bmatrix}$ of integers s.t. $n_i > 0$, $\frac{d_1}{n_1} < \cdots < \frac{d_k}{n_k}$, $d_1 + \cdots + d_k = d$, and $n_1 + \cdots + n_k = n$. The stratification has the properties

- $\mathcal{M}_{n,d}^{ss}$ has a projective "good moduli space" parameterizing "semistable bundles" up to a simple equivalence relation "S-equivalence"

- The strata S_α "deformation retracts" onto $\mathcal{M}_{n_1,d_1}^{ss} \times \cdots \times \mathcal{M}_{n_k,d_k}^{ss}$ in a suitable sense, and the latter has a projective good moduli space as well.

Classically, people mostly studied the good moduli space $\underline{\underline{M}}^{ss}_{n,d}$, because as a projective scheme it is a bit more concrete. But thinking about $\mathcal{M}_{n,d}$ and the HNS stratification is the key to many results.

④ Striking results on $\mathcal{M}^{ss}_{n,d}$

Focusing on $\mathcal{M}_{2,d}$ for simplicity. Two main results which we are heading towards:

1) Atiyah-Bott formula

$$P_t(\mathcal{M}^{ss}_{2,d}) = P_t(\mathcal{M}_{2,0}) - \sum_{k>\frac{d}{2}} t^{\#k} P_t(\mathcal{M}^{ss}_{1,k}) P_t(\mathcal{M}^{ss}_{1,d-k})$$

$(1+t)^{2g}(1+t^3)^{2g}$ $((1+t)^{2g})^2$

$k > \frac{d}{2}$

$$= \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - \sum_{k>0} t^{\#_k} \left(\frac{(1+t)^{2g}}{1-t^2}\right)^2$$

where $P_t(-) := \sum_{i \geq 0} t^i \dim H^i(-;\mathbb{Q})$ is the Poincaré polynomial, and $\#_k = 2k - d + g - 1$.

Amazing cancellation: when d odd, $P_t(\mathcal{M}^{ss}_{2,d}) \cdot (1-t^2)$ is a <u>polynomial</u>.

2) Verlinde formula: There is a unique positive generator* $L \in \operatorname{Pic}(\mathcal{M}^{ss}_{2,0})$. The Verlinde formula says $(g>1)$

$$\dim H^0(\mathcal{M}^{ss}_{2,0}, L^k) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g}$$

$$H^i(\mathcal{M}^{ss}_{2,0}, L^k) = 0 \text{ for } i > 0$$

Amazing that this is an <u>integer</u>.

(5) GIT, Kirwan surjectivity

Before we reach that material, though we will spend first half of course covering the basics of geometric invariant theory.

↳ This theory provides stratifications of a quotient stack X/G which are "toy models" for the HNS stratification "toy models" for the HNS stratification of $\mathcal{M}_{n,d}(\mathbb{C})$ $\rightsquigarrow$ Kirwan surjectivity theorem, etc.

$\hookrightarrow$ GIT is a very useful theory, with many applications of its own.

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1. Discussion of algebraic stacks, with an emphasis on examples
2. Geometric invariant theory, the kempf-Ness stratification, and good moduli spaces
3. The differential geometric perspective on GIT and the Kirwan-Ness theorem
4. The moduli of G-bundles over a curve and the relationship with Yang-Mills theory and loop groups
5. The verlinde formula, its deformations and generalizations

From <http://math.columbia.edu/~danhl/#teaching>

Plan for course:

1) will post sometimes exercises, two kinds: 1) Prove general

1) will post exercises, two kinds: 1) Prove general facts $\rightsquigarrow$ means find the proof in literature & summarize. 2) work out examples of general claims & machinery (especially in GIT section)

2) Will be compiling notes/references

3) Course philosophy $\rightarrow$ varying levels of detail, some things black-boxed when I can.

Teaching concepts vs. techniques $\rightsquigarrow$ techniques are cheap (although also crucial).